Entropy Coding for Networked Controlled Systems

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***Abstract*— The aim of this paper is to explore yet another variant of the Delta-Modulation Coding structure to improve data transmission efficiency in the context of Networked Controlled Systems. High compression rates can only be reached by the use of entropy coding. Entropy coding assigns some probability distribution to the events. In that way, the mean code length can be improved. The paper studies several issues resulting from this type of coding design and assets the stability properties needed for this type of coding to operate properly.**

***Index Terms*— Entropy coding, Networked controlled systems, NCS, quantized systems.**

I. INTRODUCTION

THis paper deals with systems interconnected by a communication network where information is transmitted via a particular coding algorithm. Many of such type of control architectures have been studied in the past. Some examples are: [8], [4], [11], [12], [14], [10], [6], [1], among others.

In particular, delta modulation (Δ-M) has been used recently in this context [3] as a mean of reducing the number of transmitted bits while preserving a methodological and simple closed-form algorithm for the coding design. Delta modulation is a well-known differential coding technique used for reducing the data rate required for voice communication, see [13]. The standard technique is based on synchronizing a state predictor on emitter and receiver and just sending a one–bit error signal corresponding to the innovation of the sampled data with respect to the predictor. The prediction is then updated by adding a positive or negative quantity (determined by the bit that has been transmitted) of absolute value Δ, a known parameter shared between emitter and receiver.

We have recently investigate the closed-loop properties of the Δ-M algorithm when used in the feedback loop. Our results in [3] have suggested some modification of the original form of the Δ-M algorithm to improve the closed-loop properties when used in feedback within the context of Networked controlled systems (NCS). The results showed that the stability domain and the resulting precision of the Δ-M is limited by the position of the largest unstable pole of the system. Although this can be improved by increasing the sampling rate, or by the use of extra bits [7], both possibilities are clearly limited by the maximum permissible data transmission rate. Further studies, have also shown that it is possible to make the modulation gain adaptive so as to improve the global stability results [5].

The aim of this work is to explore yet another variant of the Δ-M structure to improve data transmission efficiency in the context of NCS. High compression rates can only be reached by the use of entropy coding. Entropy coding is a source coding that assigns some probability distribution to the events. In that way, the mean code length can be reduced, as it will be shown here.

A pre-requisite for the entropy coding strategy is to design a mechanism with the ability to quantify and to differentiate stand-still signal events, to changes in the source (level crossing detector, denoted here as φ_{LD}). For instance, this can be done by defining an alphabet where the source signal information is contained in the time interval between level crossing and in the direction of the level crossing. As suggested in [9], by assigning strings of the 2-tuple 00 to represent the time between signal level crossing, and 01 and 10 to denote the direction of level crossing, the output of the level crossing detector contains a high probability of the 0 symbol with makes it suitable for an entropy encoder to attain a "good" overall compression ratio. A fundamental difference with the classical Δ-M algorithm is that the error is coded on the basis of a 3-valued alphabet rather than a 2-valued one. The role of the entropy coding here is to render more efficient the transmission by improving in the mean number of bits per unit of time needed for the transmission.

The overall coding strategy studied here is composed of two main blocks: an event-based (EB) coding, and a variable length-block entropy (VLE)coding scheme. The overall scheme is shown in Figure 1. The paper aims at studding the closed-loop properties of such arrangement, and investigate the possible improvement in terms of compaction ration. Due to the fact that the VLE block is a distortion-less coding, and for simplicity reasons, the focus of this study is directed toward the study of the stability properties of the first coding block only.

A. Definitions

Signals are sampled on the basis of the time interval T_s, that is at $T_s, 2T_s, ...kT_s$.

- r_k: reference signal,
- x_k: system output,
- $\hat{x}_k$: estimated(reconstructed) output,
- $\tilde{x}_k$: true estimated error, $\tilde{x}_k = x_k - \hat{x}_k$,
- $\hat{\tilde{x}}_k$: approximated estimated error, obtained after reconstruction, i.e. $\hat{\tilde{x}}_k = \{\varphi_{LD}^{-1} \circ \varphi_{LD}\}(\tilde{x}_k)$, with $\varphi_{LD}^{-1} \circ \varphi_{LD} \neq 1$.

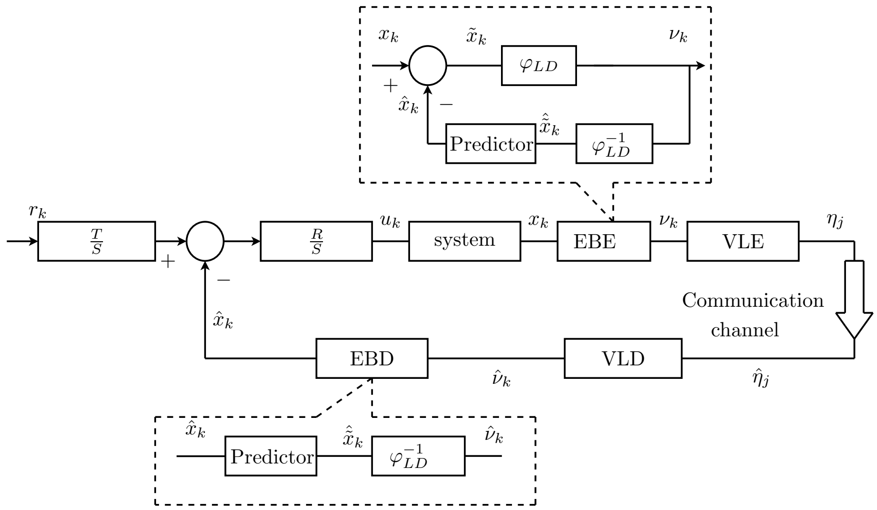

Fig. 1. Block diagram of the non uniform entropy coding in the feedback loop.

Δ: step interval used for level detection and to reconstruct $\hat{\tilde{x}}_k$,
δ_k: 3-level valued integer signal: $\{-1, 0, 1\}$
ν_k: 2-bits binary signal
η_j: variable length binary signal to be send by the channel (output of the VLE block) in asynchronous fashion, at the time instants multiples of T_s. The index j captures this asynchronism.
u_k: control input

B. Assumptions

The hypothesis used in the results presented in this paper, are the following:

- The transmitted information is binary
- Only encoder-to-decoder information transmission is allowed (feedback between decoder to encoder is forbidden),
- Reliable noiseless channel transmission is considered (no data lost, or information distortion, no transmission delays). See [2] for the treatment of transmission delay in this context.

II. Problem set up

We consider the following SISO discrete-time linear system (possible unstable), of the form,

$$x_k = \frac{B(q^{-1})}{A(q^{-1})} u_k \tag{1}$$

together with a RST controller,

$$u_k = \frac{R(q^{-1})}{S(q^{-1})} \left\{ \frac{\gamma}{T(q^{-1})} r_k - \hat{x}_k \right\} \tag{2}$$

where r_k is the reference, $\hat{x}_k$ is the estimated of the system output x_k, and $R(q^{-1}), S(q^{-1}), T(q^{-1})$ are the control polynomials in the delay operator q^{-1}. They also satisfy:

$$T = RB, \quad SA + RB = A_{cl}, \quad \gamma \triangleq A_{cl}(1)$$

with A_{cl} being the stable closed-loop polynomial, and γ the static gain needed to reach unitary zero-frequency gain. For simplicity, we will omit the use of the argument (q^{-1}) when not explicitly needed.

The coding process consists in: 1) encoding the system output x_k, 2) transmitting the coding sequence through the communication channel, and 3) decoding the received information to produce the estimated $\hat{x}_k$. The complete sequence can be seen as estimation process.

a) Forced synchronization under asynchronous transmission: The time basis for the system and the controller description is defined with respect to T_s. However, due to the variable length characteristic inherent to the entropy coding, the transmission of the coded signal ν_j is then done asynchronously at multiples of T_s. For instance, for a choice of a VLE coding of length $N = 3$, the coded signal can be sent either at; T_s, or $2T_s$, or $3T_s$ as shown in Table I. The index j captures this asynchronism.

It is assumed here that when the receiver does not receive information (this may happen if the run sequence include a stand-still event for some k) the receiver hold the last-received value $\hat{\eta}_{k-1}$ until a new change of level is detected. By this mechanism, the signals at the receiver can be re-synchronized to the time basis T_s. This is the reason why the control formulation is stated in a discrete-time synchronous representation with the sole index k, as described next.

b) Nominal closed-loop transfer function: Assume a perfect transmission process (i.e. $\hat{x}_k \equiv x_k$), then the control law (2) gives the following nominal closed-loop relation,

$$x_k = \frac{\gamma}{A_{cl}(q^{-1})} r_k$$

c) Perturbed closed-loop transfer function: Consider the case of interest where information is transmitted by the channel and quantized, i.e. $\hat{x}_k \neq x_k$. Then, the error transfer function is

$$x_k = \frac{\gamma}{A_{cl}(q^{-1})} r_k + W(q^{-1})\tilde{x}_k$$

where $\tilde{x}_k = x_k - \hat{x}_k$ is the estimation error, and $W = BR/A_{cl}$. As A_{cl} defines a stable polynomial, the output x_k is kept bounded as long as $\tilde{x}_k$ is bounded as well.

The problem is then to design the coding process that defines the output $\hat{x}_k$ preserving closed-loop properties. This process is described next.

III. Coding Process

The coding (encoding/decoding) process is composed of several steps, described by the following operations:

$$x_k \underbrace{\mapsto}_{EBE} \nu_k \underbrace{\mapsto}_{VLE} \eta_k \underbrace{\Rightarrow}_{channel} \hat{\eta}_k \underbrace{\mapsto}_{VLD} \hat{\nu}_k \underbrace{\mapsto}_{EBD} \hat{x}_k$$

As shown in Figure 1, the encoder (respectively the inverse decoder) operation is composed of two separate blocks:

- The event-based encoder EBE (respectively, event-based decoder EBD). This block maps $x_k \mapsto \nu_k$ (respectively, the decoder maps $\hat{\nu}_k \mapsto \hat{x}_k$). This block includes a level detector φ_{LD}, and a model-based predictor, MBP, similar to the one proposed in [3], and
- A variable length entropy encoder VLE (respectively, variable length decoder VLD) mapping the binary signal ν_k (run sequence, see TableI) to the N-bits η_j (respectively, the decoder maps $\hat{\eta}_j \mapsto \hat{\nu}_k$). This block includes the synchronization process described before.

A. Description of the Event-based Coding EBC

Elements composing the Event-based Coding EBC are the level detector φ_{LD}, and the model-based predictor.

1) The Level Detector: The operation principe of the level detector is shown in Figure 2. The signal detection levels are uniformly spaced by the quantum Δ. The level detector device produces a signal (identified by $'01'$ or $'10'$) whenever a level crossing takes place, and a $'00'$ if the signal remains within the level. While two symbols are used to characterize the level changes, one more symbol can be used to quantize time intervals. Then 01 indicates upward crossing, 10 downward crossing, and 00 is used to code the time-interval between crossing.

To illustrate this consider the example of Figure 2, see [9]. We assume that uniform samples are taken every time T_s, then m samples are taken in the time interval $T_i = t_i - t_{i-1}$ before a cross level takes place. As two levels (upward) crossing happen within this interval, the binary representation of this situation by the level crossing detector produce the following signal,

$$01, \underbrace{00, 00, \ldots 00}_{m-pairs}, 01$$

This sequence has then high probability of $0'$s, and thus suited for entropy coding.

To make this process operational, we introduce the operator $\varphi_{LD} : \tilde{x}_k \mapsto \nu_k$, which takes the error signal, $\tilde{x}_k$, and codes the output ν_k into a 3-valued one $\delta_k \in \{-1, 0, 1\}$. That is:

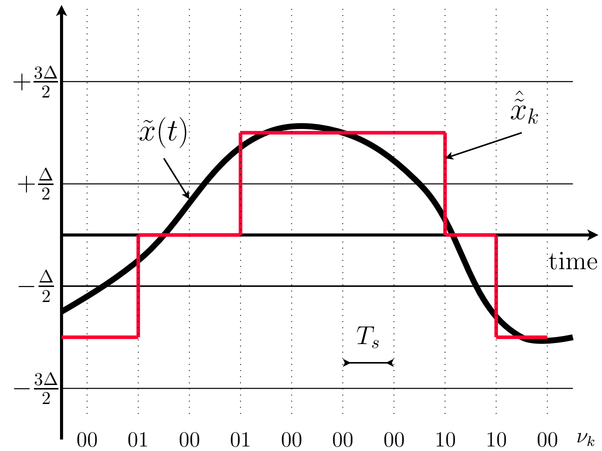

Fig. 2. Illustration of the level detector working operation principe.

$$l_k = \left\lfloor \frac{\tilde{x}_k}{\Delta} - \frac{1}{2} \right\rfloor$$

$$\delta_k = \begin{cases} 1 \text{ if } & l_k > l_{k-1}; \text{one level is crossed upwards} \\ 0 \text{ if } & l_k = l_{k-1}; \text{signal stay at the actual level} \\ -1 \text{ if } & l_k < l_{k-1}; \text{one level is crossed downwards} \end{cases}$$

with Δ the level threshold and $\lfloor \cdot \rfloor$ the floor operator which rounds to the smaller integer.

Finally, the 3-valued signal δ_k is transform into a 2-bits binary number $\nu_k \in \{00, 01, 10\}$, by the following operation.

$$\nu_k = \begin{cases} 00 & if \quad \delta_k = 0 \\ 01 & if \quad \delta_k = -1 \\ 10 & if \quad \delta_k = 1 \end{cases}$$

The combination ‘$11'$ is not used in this process.

2) The model-based predictor: Has the role to estimate (reconstruct) the encoded signal x_k, namely $\hat{x}_k$, from the 2-bits binary signal ν_k. It is composed of:

- *The inverse of the level detector:* $\varphi_{LD}^{-1} : \nu_k \mapsto \delta_k \mapsto \hat{\tilde{x}}_k$, which equations are:

$$\delta_k = \begin{cases} 0 & if \quad \nu_k = 00 \\ -1 & if \quad \nu_k = 01 \\ 1 & if \quad \nu_k = 10 \end{cases}$$

  and,

$$\hat{\tilde{x}}_k = \hat{\tilde{x}}_{k-1} + \Delta \cdot \delta_k$$

  Due to the quantization, this map does not describe an “exact inverse operator” as it will be explained latter.
- *The predictor.* The model-based predictor, as its name indicated, uses the target closed-loop model as a basis for its design. The predictor is a dynamic operator mapping the “reconstructed” error $\hat{\tilde{x}}_k$ to the “reconstructed” state $\hat{x}_k$. Its structure is inspired by our previous works in [3], [5], and also in [6]. The predictor is a dynamic linear discrete-time operator that maps the output of the

inverse level detector, to the signal prediction $\hat{x}_k$. Its structure depends upon the particular control used (state feedback or output feedback). For instance, for the RST-control discussed here, it has the following form:

$$\hat{x}_k = W\left[\frac{\gamma}{T} r_k + \hat{\tilde{x}}_k\right], \quad W \triangleq \frac{BR}{A_{cl}} \tag{3}$$

Which results in the following error equation:

$$\tilde{x}_k = W\left[\tilde{x}_k - \hat{\tilde{x}}_k\right] \tag{4}$$

B. Description of the variable length entropy coding VLE

As mentioned in the introduction, high compression rates can only be reached by the use of entropy coding. By assigning some probability distribution to the events, the mean code length can be optimized. Run-length codes[1], are a class of variable-length codes that are sub-optimal (when compared to the Huffman code), but have the advantage of avoiding buffering at the decoder side, and therefore reducing data transmission latency. An example used in [9] is described in Table I.

The VLE of length N can be described as a memory map from ν_k to η_j. The VLE block contains a buffer of dimension l that stores information of past values of ν_k. The buffer dimension l depends on N. The buffer information is used to build a run sequence resulting from the composition of $\nu_{k-l}, ..., \nu_k$. This sequence, which is a variable-length binary signal, is named the "run sequence" used to produce the output η_j. The variable-length nature of this sequence introduces a variable latency (asynchronous output) which is multiple of T_s. An example of a coding scheme of block length $N = 3$ is shown below.

Run sequence $\nu_{k-2}, \nu_{k-1}, \nu_k$	Transmission period (sec)	Output η_j $(N = 3)$
01	T_s	000
10	T_s	001
00 01	$2T_s$	010
00 10	$2T_s$	011
00 00 01	$3T_s$	100
00 00 10	$3T_s$	101
00 00 00	$3T_s$	110
unused	-	111

TABLE I

RUN-LENGTH ENCODING.

Assuming that the coding sequence is independent and identically distributed, and that the upward crossing frequency equals the downwards crossing frequency, i.e.

$$p = P(00), \quad P(01) = P(10) = \frac{1}{2}(1-p)$$

where $p \in [0, 1]$ is the probability to have an stand-still event. According to [9], the mean coding length, C_L, of this scheme is:

$$C_L = 2\frac{1 - p^{(2^{(N-1)}-1)}}{1-p} bits$$

[1]Class of coding strategy that can decode information instantaneously.

which has the limiting value: $\lim_{p\to 1} C_L = 2^N - 2$. The compaction ratio is $C_R = \frac{C_L}{N}$.

Let R_ν be the transmission rate when the 2-bits signal ν_k is sent synchronously at each T_s without using an entropy code. This rate is given by $R_\nu = 2[\text{bits}/T_s]$.

When the VLE coding is used assuming a probability p to the stand-still event, the mean transmission rate R_m [bits/T_s] associated to a code of length N is

$$R_m = \frac{\text{number of bits sent}}{\text{mean transmission period}} = \frac{N}{T_m}$$

where $T_m = \frac{C_L}{2}[T_s]$. This gives

$$R_m = \frac{2N}{C_L} = R_\nu \frac{N}{C_L}$$

From this expression we can evaluate the potential improvement due to the entropy coding in terms of the mean rate needed for this scheme to work. In particular if p is large enough so that $\frac{N}{C_L} < 1$, then the VLE scheme will provide lowest rates. Table II compares these rates for different values of N and p. Note that very low rates are required for large N and probabilities close to one(the bit rate is reduced by an order of magnitude at $p = 1$).

N	R_m (with VLE) $p = 0$	$p = 0.5$	$p = 1$	R_ν (without VLE)
3	3	1.7	1.00	2
4	4	2.0	0.57	2
5	5	2.5	0.33	2
6	6	3.0	0.19	2

TABLE II

MEAN TRANSMISSION RATE R_m FOR $p = 0$,$p = 0.5$ AND $p = 1$

IV. ERROR SYSTEM AND STABILITY CONDITIONS

Following the assumptions maid in this paper (lossless channel transmission), we then have that $\nu_k = \hat{\nu}_k$, and that $\delta_k = \hat{\delta}_k$. In this case binary variables are not needed, and hence error equation can be described by real variables only.

A. Error equations

Introducing the following error definitions:

- $e_k = x_k - \frac{\gamma}{A_{cl}} r_k$: the tracking error,
- $\tilde{x}_k = x_k - \hat{x}_k$: the prediction error, and
- $\varepsilon_k = \tilde{x}_k - \hat{\tilde{x}}_k$: error due to the non exact inverse mapping of the level detector, i.e. due to the map $\varphi_{LD} \circ \varphi_{LD}^{-1}$.

we have the closed-loop error system:

$$e_k = W(q^{-1})\tilde{x}_k \tag{5}$$
$$\tilde{x}_k = W(q^{-1})\varepsilon_k \tag{6}$$

with $W = BR/A_{cl}$ being the stable operator defined previously. Note that the $\varepsilon_k = \varepsilon_k(\tilde{x}_k)$, and thereby the above error equation can be seen as two systems in cascade, i.e. the output of the autonomous system (6) is the input of the stable system (5). For stability purposes it is thus sufficient to demonstrate the stability properties of the sub-system (6).

Note that ε_k writes as:

$$\begin{aligned}\varepsilon_k &= \tilde{x}_k - \hat{\tilde{x}}_k \\ &= \tilde{x}_k - \varphi_{LD} \circ \varphi_{LD}^{-1}\{\tilde{x}_k\} \\ &= \tilde{x}_k - \tilde{\varphi}_{LD}\{\tilde{x}_k\}\end{aligned}$$

where $\tilde{\varphi}_{LD} \stackrel{\triangle}{=} \varphi_{LD} \circ \varphi_{LD}^{-1} : \tilde{x}_k \mapsto \hat{\tilde{x}}_k$. Note that this map is dynamic, defined by the following relation:

$$\hat{\tilde{x}}_k = \hat{\tilde{x}}_{k-1} + \Delta \cdot \delta_k \tag{7}$$

with $\delta_k = f(\tilde{x}_k)$ as defined before. The sub-system (6)-(7) can be then seen as a feedback system,i.e.

$$\begin{aligned}\tilde{x}_k &= W(q^{-1})\varepsilon_k \\ &= W(q^{-1})\left(\tilde{x}_k - \hat{\tilde{x}}_k\right) \\ &= W(q^{-1})\left(\tilde{x}_k - \frac{\Delta}{1-q^{-1}}\delta_k(\tilde{x}_k)\right)\end{aligned}$$

Ideally we would like that the map $\tilde{\varphi}_{LD}$ be a linear map with unitary gain. This ideal goal is hampered by several factors:

- unknown initial conditions of $\tilde{x}_0$,
- badly chosen T_s, and Δ, and
- chattering in the neighborhood of the quantum Δ.

In particular, large sampling times T_s, and too small quantum Δ may results in signal variation of more than one level, which may leads to unrecovered bias in the estimated, leading to potential instabilities for unstable open-loop systems. This stabilities issues are analyzed next for a system of one dimension.

B. Stability properties

Consider the stabilization problem ($r = 0$) of the following simple unstable system $\frac{B(q^{-1})}{A(q^{-1})} = \frac{bq^{-1}}{1-aq^{-1}}$, with $2 > |a| > 1$, and the control law $u = kx_k$. Let $1 > a_c > 0$ be the desired closed loop poles, the required gain to reach such closed-loop specification is $k = (a-a_c)/b$. This particular choice leads to the error equations (5)-(6) with $W(q^{-1}) = \frac{(a-a_c)q^{-1}}{1-a_c q^{-1}}$. Due to the cascade structure of such error equation arrangement, we mainly will concentrate in the equation (6) which captures most of the difficulties. To this aim we will concentrate on the following set of equations, which describes the error feedback interconnection.

$$\tilde{x}_{k+1} = a_c\tilde{x}_k + (a-a_c)\varepsilon_k, \qquad \varepsilon_k = \tilde{x}_k - \hat{\tilde{x}}_k \tag{8}$$

$$\hat{\tilde{x}}_k = \hat{\tilde{x}}_{k-1} + \Delta \mathrm{sign}\left(l_k - l_{k-1}\right), \quad l_k = \left\lfloor \frac{\tilde{x}_k}{\Delta} - \frac{1}{2} \right\rfloor \tag{9}$$

The analysis is divided into two steps:

- Rate level condition. We first derive conditions on a, and a domain B_{ρ_1} for $\tilde{x}_k$ that ensures that no more than one level change can be effectuated, i.e. $|l_k - l_{k-1}| \leqslant 1$,
- Invariance condition. Then, by a Lyapunov-like analysis we show that this domain is indeed an invariant; solutions $\tilde{x}_k$ starting in B_{ρ_1} do not leave this domain.

C. Rate level condition

We seek here to establish condition on $|\tilde{x}_k|$, $\forall k \in \mathbb{Z}^+$ such that the rate change in the level detector be at most one. To be consistent with this aim, we need to assume in the sequel that the encoder/deconder internal states are suitable initialized. That is, $\hat{x}_0$, and l_0 are such that: $\varepsilon_0 < \Delta/2$, and $\hat{\tilde{x}}_0 = \Delta l_0$ at $k = 0$.

Lemma 1: Consider unstable systems limited by the relation $a < 2 + a_c < 3$, and let define the compact set, B_{ρ_1}, as:

$$B_{\rho_1} = \{\tilde{x}_k : |\tilde{x}_k| < \rho_1\}, \quad \rho_1 = \frac{(1 - \frac{(a-a_c)}{2})}{1-a_c}\Delta$$

with $\rho_1 > 0$. Then for all $|\tilde{x}_k| \in B_{\rho_1}$ the following holds, $\forall k \in \mathbb{Z}^+$:

$i)$ $|\tilde{x}_k - \tilde{x}_{k-1}| < \Delta$,

furthermore, $i)$ implies the following two equivalent inequalities:

$ii)$ $|l_k - l_{k-1}| \leqslant 1$

$iii)$ $|\varepsilon_k| \leqslant \Delta/2$

Proof: Let us start with the last part of this result, i.e. $(i) \Rightarrow \{(ii) \Leftrightarrow (iii)\}$. By inspection, it is easy to see that $(i) \Rightarrow (ii)$; if the rate of change of $\tilde{x}_k$ is strictly smaller than Δ then the level change is limited, by definition, to a maximum one. In turn, $(ii) \Rightarrow (iii)$, results from the following arguments.

If the initialization condition $\hat{\tilde{x}}_0 = \Delta l_0$ and (ii) holds, then we have that $\hat{\tilde{x}}_k = \Delta l_k$, $\forall k \in \mathbb{Z}^+$. Then it follows that the error between the true estimation error, and the reconstructed one is bounded by the amount $\Delta/2$, i.e.

$$\begin{aligned}\varepsilon_k = \tilde{x}_k - \hat{\tilde{x}}_k = \tilde{x}_k - \Delta l_k = \tilde{x}_k - \left\lfloor \frac{\tilde{x}_k}{\Delta} - \frac{1}{2} \right\rfloor \\ = \Delta\left(\frac{\tilde{x}_k}{\Delta} - \left\lfloor \frac{\tilde{x}_k}{\Delta} - \frac{1}{2} \right\rfloor\right) \leqslant \frac{\Delta}{2}.\end{aligned}$$

Inversely, if (iii) holds, it is easy by inspection to see that (ii) is true.

Now the first part of the lemma (i) is proven. From (8) we have

$$|\tilde{x}_{k+1} - \tilde{x}_k| < (1-a_c)|\tilde{x}_k| + (a-a_c)|\varepsilon_k|$$

assume for the moment that (iii) holds, then condition for fulfilling $i)$ is that

$$|\tilde{x}_{k+1} - \tilde{x}_k| < (1-a_c)|\tilde{x}_k| + (a-a_c)|\Delta/2 < \Delta$$

or equivalently that

$$|\tilde{x}_k| < \frac{(1 - \frac{(a-a_c)}{2})}{1-a_c}\Delta = \rho_1 \tag{10}$$

For this expression to be valid (i.e. $\rho_1 > 0$) , we require the condition $a < 2 + a_c < 3$. To conclude, assume that (10) holds independently to $(i) - (iii)$ (as it will be shown latter in the next section). Now if (i) holds we just show that this implies (iii), which in turn and together with (10), implies (i). ∎

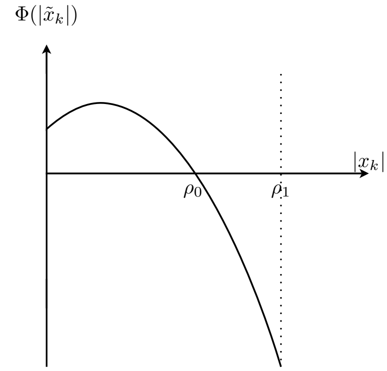

Fig. 3. $\Phi(|\tilde{x}_k|)$.

D. Invariance condition

Question here is to find under which conditions we can ensure the invariance of the set B_{ρ_1}. This invariance condition is clearly needed to preserve the rate level condition mentioned previously. We assume initially that the observation error is inside that set, and we look for the condition such that this signal does not leaves B_{ρ_1}.

To this aim, consider the Lyapunov function $V_k = \tilde{x}_k^2$, and its rate variation $\nabla V_k \triangleq \tilde{x}_{k+1}^2 - \tilde{x}_k^2$, i.e.

$$\begin{aligned}\nabla V_k &= (a_c^2 - 1)|\tilde{x}_k|^2 + 2(a - a_c)a_c\varepsilon_k\tilde{x}_k + (a - a_c)^2\varepsilon_k^2 \\ &\leqslant (a_c^2 - 1)|\tilde{x}_k|^2 + (a - a_c)a_c|\tilde{x}_k|\Delta + \frac{(a - a_c)}{4}^2\Delta^2 \\ &\triangleq \Phi(|\tilde{x}_k|)\end{aligned}$$

Where the last inequality is results from the hypothesis that initially we assume $\tilde{x}_k \in B_{\rho_1}$, or equivalently (see Lemma 1) that $|\varepsilon_k| \leqslant \Delta/2$.

The shape of the polynomial $\Phi(|\tilde{x}_k|)$ is shown in Figure 3, this function has two roots, one negative and other positive. The positive root, is ρ_0, and is given by

$$\rho_0 = \frac{(a - a_c)}{2(1 - a_c)}\Delta$$

It is necessary for stability that $\rho_0 < \rho_1$, else a local stability region may not exist. It is easy to show that this condition is valid as long as $a < a_c + 1$, which is the same condition already assumed by Lemma 1.

The value of ρ_0 defines the limit (somewhat conservative) after which the function V_k, and hence the norm of x_k decrease. Below that limit the function may grow. The worst case growth in the interval $|\tilde{x}_k| \in [0, \rho_0]$, can be estimated from the relation

$$\tilde{x}_{k+1}^2 \leqslant \left(a_c|\tilde{x}_k| + \frac{(a - a_c)}{2}\Delta\right)^2 \triangleq \psi(|\tilde{x}_k|)$$

As the function $\psi(|\tilde{x}_k|)$ is convex in $|\tilde{x}_k| \in [0, \rho_0]$ its maximum is located at the extremes of this interval. The worst case growth is then defined by the following relation:

$$x_{max} = \max\left\{\sqrt{\psi(0)}, \sqrt{\psi(\rho_0)}\right\} = \Delta\frac{(a - a_c)}{2(1 - a_c)}$$

Finally, the set B_{ρ_1} is invariant if $x_{max} < \rho_1$, i.e.

$$\Delta\frac{(a - a_c)}{2(1 - a_c)} < \Delta\frac{(1 - \frac{(a - a_c)}{2})}{1 - a_c}$$

working out details of this inequality, it can be shown that this equality holds if $a - a_c < 1$, for all $a_c \in (0, 1)$. Note that this is a stronger condition than the one in Lemma 1 as it is derived from a more conservative analysis.

The following theorem summarizes the main result.

Theorem 1: Assume that the coding algorithm is initialized such that $\hat{x}_0$, and l_0 are such that: $\varepsilon_0 < \Delta/2$, and $\hat{\tilde{x}}_0 = \Delta l_0$. Consider system satisfying $a - a_c < 1$, with initial condition in the set $\tilde{x}_0 \in B_{\rho_1}$. Then:

- $\tilde{x}_k \in B_{\rho_1}$, $\forall k \in \mathbb{Z}^+$,
- $\exists k_0 : |\tilde{x}_k| \leqslant \rho_0$, $\forall k \geqslant k_0$, and
- $\lim_{k\to\infty} d(x_k, \mathcal{B}_\beta) = 0$.

where $d(x_k, \mathcal{B}_\beta)$ is the minimum Euclidean distance from x_k to any point within the ball

$$\mathcal{B}_\beta := \{x \in \mathbb{R} : \|x\| < \beta\},$$

and β is a constant that depends on ρ_0, and on the infinite norm of $W(q^{-1})$.

Proof: The first two statements follow from the previous analysis, the last statement result from equation (5), i.e; $|x_k| \leqslant ||W|| \cdot |\tilde{x}_k|$. Details for the derivation of this property are similar to the ones used in [3], and [5]. ■

V. Simulation evaluation

We consider the following simple system

$$\frac{B(q^{-1})}{A(q^{-1})} = \frac{bq^{-1}}{1 - aq^{-1}} \tag{11}$$

The controller is: $u_k = -k\hat{x}_k + \gamma r_k$ obtained from the closed-loop specification given by $A_{cl} = (1 - a_{cl}q^{-1})$, $k = \frac{a - a_{cl}}{b}$ and $\gamma = 1 - a_c$. Parameter used in simulations are: $a = 1.1$, $b = 1$, $a_c = 0.9$, $T_s = 0.05$ (sec), $\Delta = 0.02$, $x_0 = 0$ and $\hat{x}_0 = -0.01$ so $\tilde{x}_0 = 0.01$.

The purpose of this section is to evaluate in simulation the proposed algorithm and to discuss several issues concerning the algorithm implementation. Among these, we have: initial synchronization, effect of the quantum Δ and the sampling time T_s, among others.

A. Initial conditions

Initial conditions of the predictor at the encoder side $\hat{x}_0$ need to be synchronized with initial condition predictor at the decoder side. This requires a specific initialization procedure that send this initial information before the coding process is triggered.

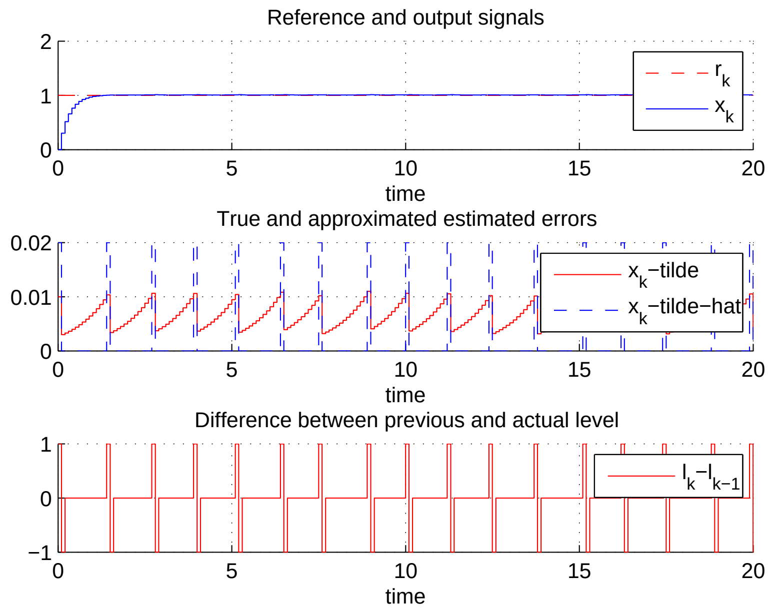

Fig. 4. Simulation results with $\Delta = 0.02$; $T_s = 0.05$ yielding $|l_k - l_{k-1}| \leqslant 1$. Output and reference (upper), $\tilde{x}$ vs.$\hat{\tilde{x}}$ (middle), and $l_k - l_{k-1}$ (lower)

B. *Quantum* Δ *and sampling time* T_s

The value of Δ has an important impact on the quality of the estimates and the system stability. Larger values of Δ will enlarge the attraction domain, but it will degrade the estimated quality. Inversely, small values for Δ will improve the reconstructed signal quality, but it will diminish the local attraction domain.

The impact of T_s can be seen from the necessary condition for stabilization, i.e. $a = e^{\alpha T_s} < 2$, where α is the open-loop pole of the continuous system. For unstable systems, T_s is limited by the relation $T_s < \ln(2)/\alpha$. The more the system is unstable the smaller need to be T_s and hence the transmission bit rates need to be increased.

The Figure 4, shows a simulation where parameters are selected to satisfy the stability conditions. As a consequence, the change of level is limited to one.

C. *Event distribution and its impact in the mean transmission rate*

Note that practically all parameters of the control scheme affects the spectrum of the evolution of δ_k, and in particular the Δ, and T_s, but also the magnitude of open-loop unstable poles. Figure 5 shows the resulting histogram of δ_k for two simulations with two different values of a.

The results show that for small values of a the frequency spectrum of δ_k is reduced, and hence the event $\delta_k = 0$ has higher probability to occur. We recall that distributions with high roll-off will be benefic for data compression, as illustrated by the VLE algorithm. We can then conclude that open-loop instable systems with a high degree of instability are less adapted for entropy coding. The resulting compression ratio are reported in Table III. Higher compression rates

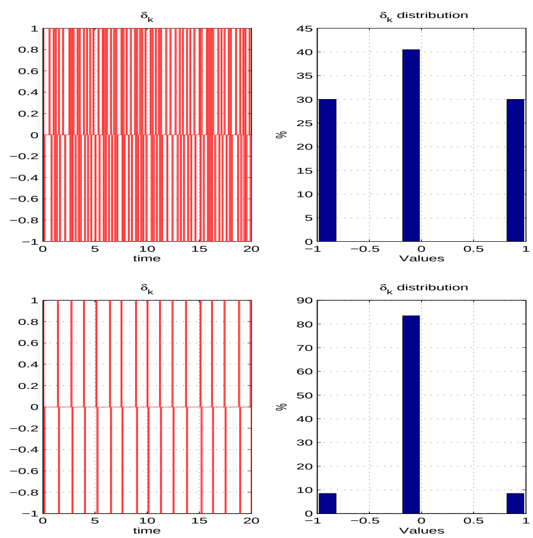

Fig. 5. Simulations with $\Delta = 0.02$; $T_s = 0.05$; $a = 1.6$ (upper), $\Delta = 0.02$; $T_s = 0.05$; $a = 1.1$ (bottom), Time evolution of δ_k (left), histogram of events (right)

are thus obtained for the cases where p is higher, i.e. the bottom figures in Fig.5.

From the Table III we can see that for $a = 1.1$, the best choice is a VLE coding of length $N = 4$, or $N = 5$, whereas for the system with $a = 1.6$ only the VLE with $N = 3$ improves over the one without entropy coding.

VI. Conclusions

In this paper we have investigated the possibility to use entropy coding in the context of networked controlled sys-

N	R_m (with VLE)		R_ν (without VLE)
	$p = 0.4$; $a = 1.6$;	$p = 0.8$ $a = 1.1$	
3	1.92	1.23	2
4	2.40	1.01	2
5	3.00	1.03	2
6	3.60	1.20	2

TABLE III

MEAN TRANSMISSION RATES FROM DATA SHOWN IN FIG. 5.

tems. The main motivation has been to explore the benefits in terms of mean transmission rate. In particular we have analyzed the case of variable-length encoder VLE, which in spite of its sub-optimality do not require buffering at the decoder side, and hence reduce latency.

We have shown that the scheme results in a local stable system with an attraction domain and a final precision depending on the value of the level granularity Δ. Finally we have demonstrated by simulation the benefits that can be reached by using this type of coding, and for a given system with a fixed scenario, how to select the length of the VLE code. From this study it results also that system with a low degree of instability (eigenvalues closed to one), the optimal value of N is in general larger than for system with high degree of instability.

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